Yet another QP solver for robotics and beyond

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Convex QP problem

Convex QP problem

Setting

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H x + g^T x$$

$$\begin{cases} Ax = b \\ Cx - d \leq 0 \end{cases}$$

$H \in \mathcal{S}_+(\mathbb{R}^n),\ A \in \mathbb{R}^{n_{eq} \times n},\ C \in \mathbb{R}^{n_{in} \times n}$

Links with motion planning

Aim: control a given dynamic

$$x_{t+1} = Ax_t + Bu_t$$
$$\begin{cases} Cx_t = d_t \\ \ldots \end{cases}$$

Classic tool: LQR command

$$\min_u \sum_{t=0}^{T-1} \frac{1}{2}u_t^T R u_t + \frac{1}{2}x_t^T Q x_t + \frac{1}{2}x_T D x_T$$
$$\begin{cases} x_{t+1} = Ax_t + Bu_t \\ Cx_t = d_t \\ \ldots \end{cases}$$

Specifications for robotics

A “good” solver should be

- fast,
- accurate,
- numerically robust,
- capable to deal with unfeasible QPs.

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Current solver approaches

State-of-the-art convex QP solvers

- Active set methods : QPoases, Quadprog,
- Penalization methods
  - Interior Point methods : Gurobi, Mosek,
  - Augmented Lagrangian methods : OSQP, QPALM.

Global minimum necessary and sufficient conditions

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H x + g^T x \quad \begin{cases} Ax = b \\ Cx - d \leq 0 \end{cases} \tag{1}$$

$H \in \mathcal{S}_+(\mathbb{R}^n)$, $A \in \mathbb{R}^{n_{eq} \times n}$, $C \in \mathbb{R}^{n_{in} \times n}$

x^* is a global minimum of problem 1 iff $\exists (y^*, z^*) \in \mathbb{R}^{n_{eq}} \times \mathbb{R}_+^{n_{in}}$

$$x^*, y^*, z^* \text{ satisfies KKT conditions} \tag{2}$$

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Our approach

Two main ideas

- "Mixing bests" of two Augmented Lagrangian based algorithms
- Using an initial guess procedure accelerating subproblems solving

Augmented Lagrangian

$$L_A(x, y_k, z_k, \mu_{eq,k}, \mu_{in,k}) := \max_{z \geq 0, y} \mathcal{L}(x, y, z) - \frac{1}{2\mu_{eq,k}} \|y - y_k\|^2$$

$$-\frac{1}{2\mu_{in,k}} \|z - z_k\|^2$$

Mixing proximal methods with selective updates

Proximal method of multiplier

Repeat

- $x_{k+1} \approx \arg\min_{x \in \mathbb{R}^n} L_A(x, y_k, z_k, \mu_{eq,k}, \mu_{in,k}) + \frac{\rho}{2}\|x - x_k\|^2$
- $y_{k+1} = y_k + \mu_{eq,k}(Ax_{k+1} - b)$
- $z_{k+1} = [z_k + \mu_{in,k}(Cx_{k+1} - d)]_+$

Bound Constrained Augmented Lagrangian

Repeat

- Find x_{k+1} s.t $\|\nabla_x L_A(x_{k+1}, y_k, z_k, \mu_{eq,k}, \mu_{in,k})\| \approx 0$
- If $\max(\|Ax_{k+1} - b\|, \|[Cx_{k+1} - d]_+\|) \approx 0$ update y_k and z_k
- Else, increase $\mu_{eq,k}$ and $\mu_{in,k}$

Current algorithm

Repeat

- $x_{k+1} \approx \arg\min_{x \in \mathbb{R}^n} L_A(x, y_k, z_k, \mu_{eq,k}, \mu_{in,k}) + \frac{\rho}{2}\|x - x_k\|^2$
  - with initial guess procedure
  - or with correction procedure
- If $\max(\|Ax_{k+1} - b\|, \|[Cx_{k+1} - d]_+\|) \approx 0$ update y_k and z_k
- Else, increase $\mu_{eq,k}$ and $\mu_{in,k}$

Augmented Lagrangian

$$L_A(x, y_k, z_k, \mu_{eq,k}, \mu_{in,k}) := \max_{z \geq 0, y} \mathcal{L}(x, y, z) - \frac{1}{2\mu_{eq,k}} \|y - y_k\|^2$$

$$- \frac{1}{2\mu_{in,k}} \|z - z_k\|^2$$

Subproblem solving method

$$\min_{x} L_A(x, y_k, z_k, \mu_{eq,k}, \mu_{in,k}) + \frac{\rho}{2} \|x - x_k\|^2 =$$

$$\min_{x} \max_{z \geq 0, y} \mathcal{L}(x, y, z) - \frac{1}{2\mu_{eq,k}} \|y - y_k\|^2 - \frac{1}{2\mu_{in,k}} \|z - z_k\|^2 + \frac{\rho}{2} \|x - x_k\|^2$$

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Results

Maros Meszaros convex QPs data set

- 138 hard QPs
- Recognised data set to benchmarks best solvers

Standard random QPs

- equality QP
- inequality QP
- constrained LQR (with equalities and inequalities)

What is measured ?

- KKT conditions satisfability

Benchmark on 60% of Maros Meszaros convex QPs

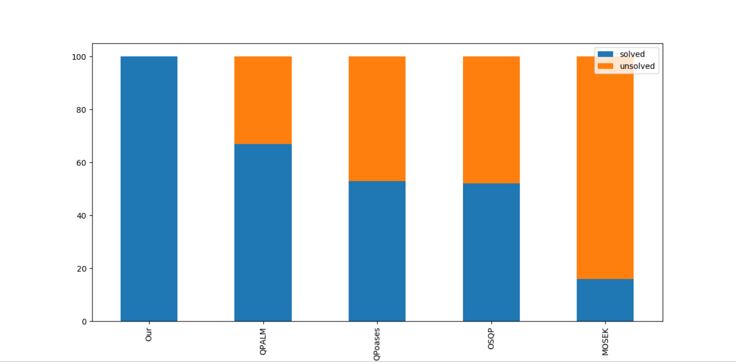

Figure: Percentage of problems solved of our approach, QPALM, OSQP, QPoases and Mosek

Benchmark on 60% of Maros Meszaros convex QPs

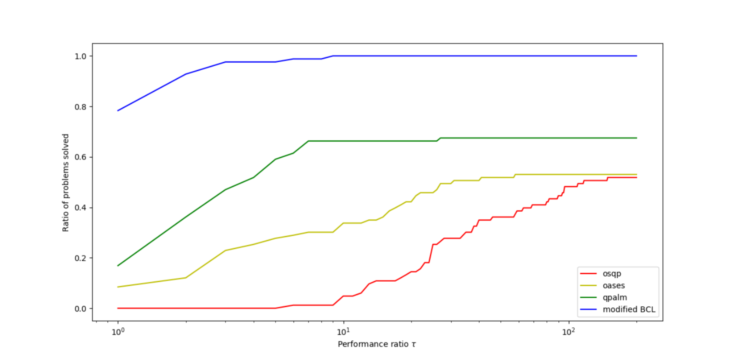

Figure: Performance profiles of our approach, QPALM, OSQP and QPoases

Benchmark on a synthetic equality QP

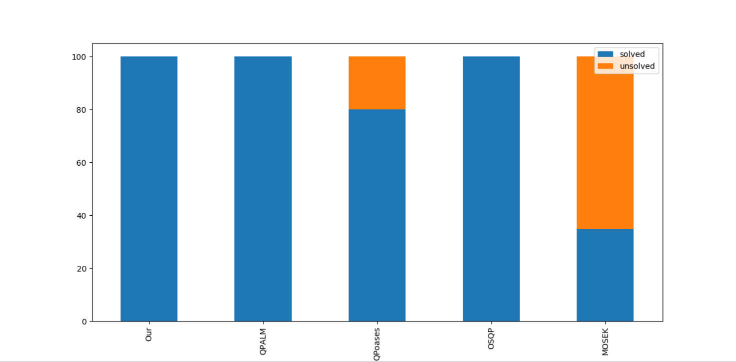

Figure: Percentage of problem solved for our approach, QPALM, OSQP, QPoases and MOSEK

Benchmark on a synthetic equality QP

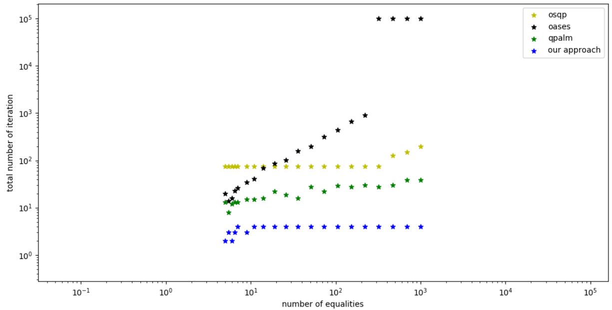

Figure: Total number of iterations of our approach, QPALM, OSQP and QPoases

Benchmark on a synthetic inequality QP

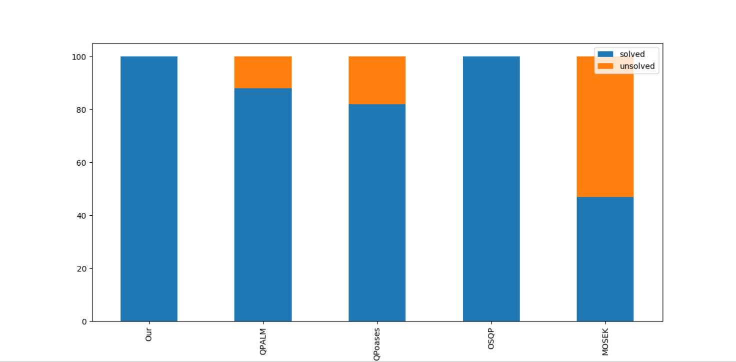

Figure: Total number of iterations of our approach, QPALM, OSQP and QPoases

Benchmark on a synthetic inequality problem

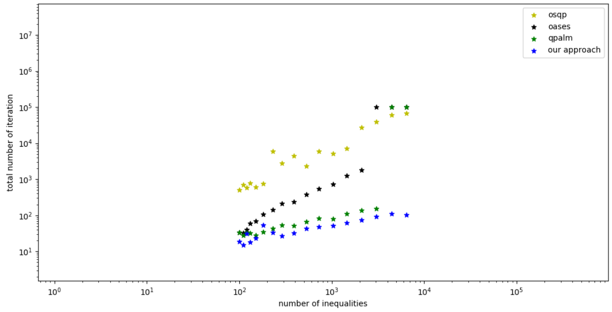

Figure: Total number of iterations of our approach, QPALM, OSQP and QPoases

Benchmark on a synthetic LQR problem

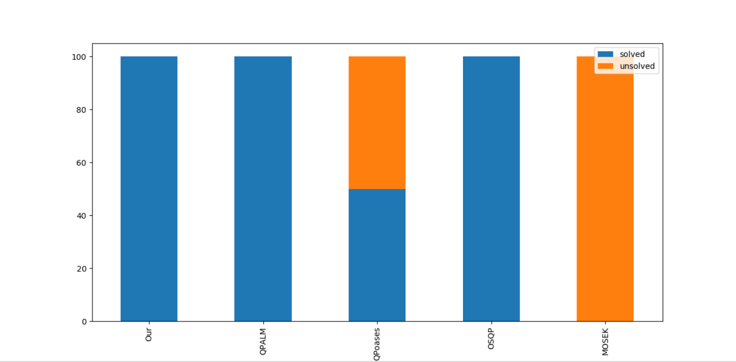

Figure: Total number of iterations of our approach, QPALM, OSQP and QPoases

Benchmark on a synthetic control problem

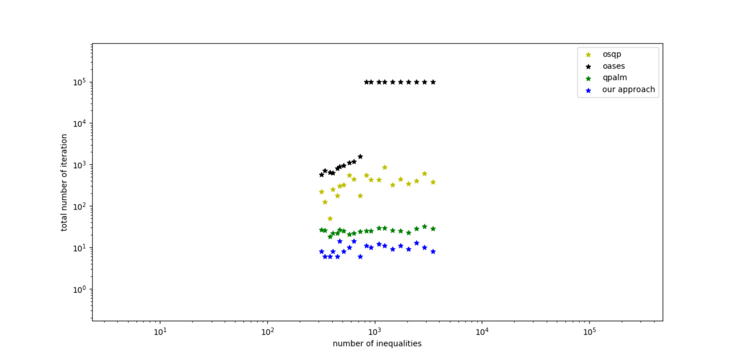

Figure: Total number of iterations of our approach, QPALM, OSQP, and QPoases

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Conclusion

Current work

- theoretical guarantees
- C++ implementation

Next studies

- Dealing with non feasible convex QPs
- Applications to LQR and ML

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Annexes

Benchmark on 60% of Maros Meszaros convex QPs

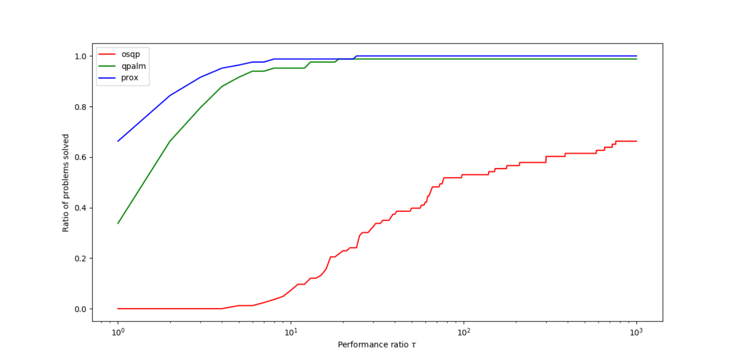

Figure: Performance profile of PROX-BCL, QPALM and OSQP

Benchmark on a synthetic inequality QP

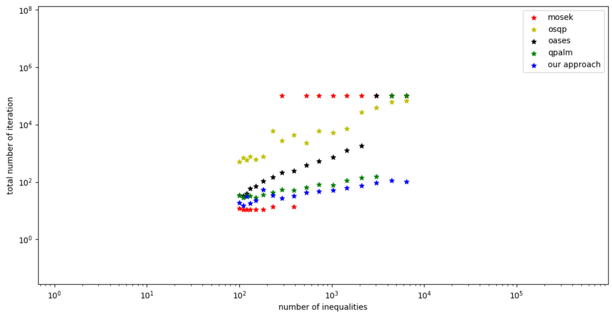

Figure: Total number of iterations of PROX-BCL, QPALM and OSQP

Benchmark on 60% of Maros Meszaros convex QPs

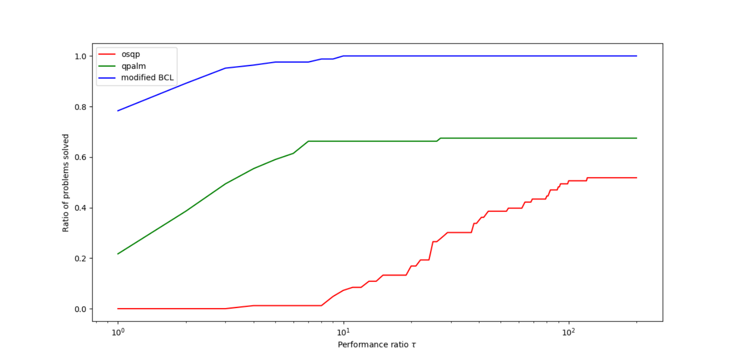

Figure: Performance profiles of our approach, QPALM and OSQP

Penalty method

Motivation : get unconstrained optimization by adding squared violated constraints to the objective.

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H x + g^T x + \frac{\mu_{eq}}{2} \|Ax - b\|_2^2 + \frac{\mu_{in}}{2} \sum_i max((Cx - d)_i, 0)^2$$

Cons

- In the idealitsic equality constrained scenario

$$(Ax_k - b)_i \approx -\frac{y_i^*}{\mu_k}$$

Exact line search

Given a step direction $(dx, dy, dz)^T$

$$\min_{\alpha \in \mathbb{R}} \|\nabla L_{I(\alpha)}(x + \alpha dx, y + \alpha dy, \max(z + \alpha dz, 0))\|_2^2$$

$I(\alpha) := \{i \in \{1, ..., n_{in}\} | z_i + \alpha dz_i > 0\}$

Other numerical considerations

- Ruiz equilibration (speed up)
- Warm starting
- exploration nodes
- ...

Intuitive idea - 70s

Find the optimal active set. Apply convex optimization techniques for convex equality constrained QPs.

Optimality condition

$x*$ is a global minimum iff $\exists(y^*, z^*)$

$$\begin{cases} Hx^* + g + A^T y^* + C_I^T z_I^* = 0 \\ Ax^* - b = 0 \\ C_I x^* - d_I = 0 \\ C_{I^c} x^* - d_{I^c} < 0 \end{cases}$$

$I := \{i \in \{1, ..., n_{in}\} | (Cx^* - d)_i = 0\}$

Active set method

Find iteratively an optimal active set. Apply convex optimization techniques for convex equality constrained QPs.

Interior point method

$$\min_x \frac{1}{2}x^T Hx + g^T x + \phi(x)/t$$
$$\text{s.t } Ax = b, t > 0$$

With $\phi(x) := -\sum_{i=1}^{m} log(-(C(x) - d)_i)$

Augmented Lagrangian method

$$L_A(x, y, z, \mu_{eq}, \mu_{in}) := \frac{1}{2}x^T Hx + g^T x$$
$$+\frac{\mu_{eq}}{2}(\|Ax-b+\frac{y}{\mu_{eq}}\|_2^2-\|\frac{y}{\mu_{eq}}\|^2)+\frac{\mu_{in}}{2}(\|[Cx-d+\frac{z}{\mu_{in}}]_+\|^2-\|\frac{z}{\mu_{in}}\|^2)$$

Intuitive idea - 70s

Find the optimal active set to apply convex optimization techniques for convex equality constrained QPs.
Repeat

- Pick subset $\mathcal{W}_k$ of $\{1, ..., n_{in}\}$
- Find $x_{k+1} = \arg\min q(x)$ subject to $C_{i,}^T x = d_i, \forall i \in \mathcal{W}_k$
- If x_{k+1} does not solve QP, adjust $\mathcal{W}_k$ to form $\mathcal{W}_{k+1}$

Cons

- Gradient of the constraints must be linearly independent (degeneracy). Provokes $\mathcal{W}_k$ cycling (slow)
- Not really robust to scale
- Worst complexity : exponential

Intuitive ideas - 90s

- Replace inequality constraints by a twice continuously differentiable penalization function,
- Solve the new problem with convex optimization techniques for convex equality constrained QPs.

Approximation via logarithmic barrier

$$\min_x tq(x) + \phi(x)$$

$$\text{s.t } Ax = b$$

With $\phi(x) := -\sum_{i=1}^{m} log(-(C(x) - d)_i),\ t > 0.$

Pros and cons

- Pros : Robustness
- Cons :
  - > Not best precision,
  - > Not best speed (no possible warm start).

Intuitive ideas - 2015 (0SQP)

- Introduce auxiliary variable to handle inequality : $Cx = z$
- Applying ADMM for the resulting problem

ADMM

$$\min_{x,z} f(x) + g(z)$$

$$\text{s.t } Ax + Bz = c$$

Repeat

- $x_{k+1} = \arg\min_x L_\rho(x, z_k, y_k)$
- $z_{k+1} = \arg\min_z L_\rho(x_{k+1}, z, y_k)$
- $y_{k+1} = y_k + \rho(Ax_{k+1} + Bz_{k+1} - c)$

$L_\rho(x, y, z) = f(x) + g(z) + y^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|^2$

Pros and cons

- Pros : precision
- Cons :
  - > Slow (for high precision),
  - > Less robust than IP methods.

Maros Meszaros convex QPs sizes

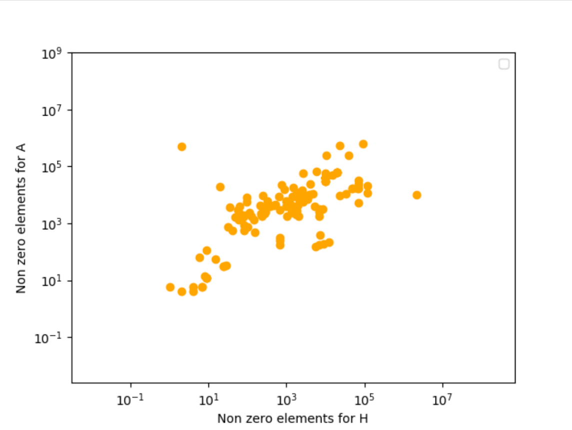

Figure: Number of non zeros elements in Maros Meszaros matrices

OSQP results on 134 problems with 10^5 iterations

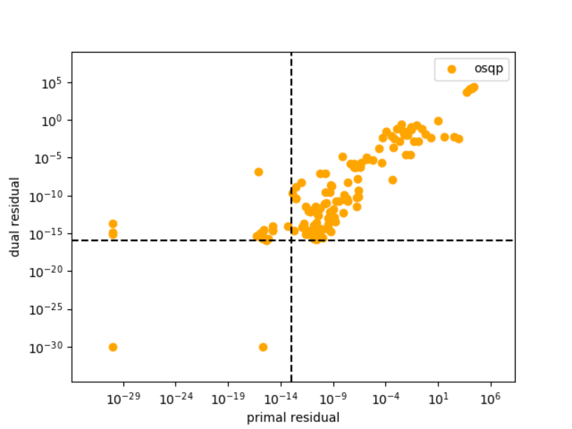

Figure: OSQP primal and dual residuals on first 134 Maros problems

Benchmark on 60% of Maros Meszaros convex QPs

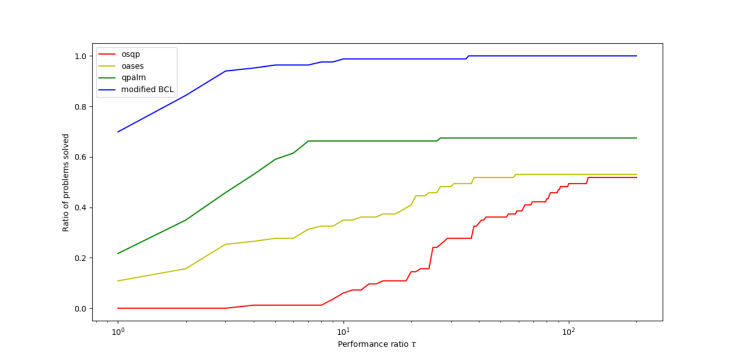

Figure: Performance profiles of our approach, QPALM, OSQP and QPoases

Benchmark on 60% of Maros Meszaros convex QPs

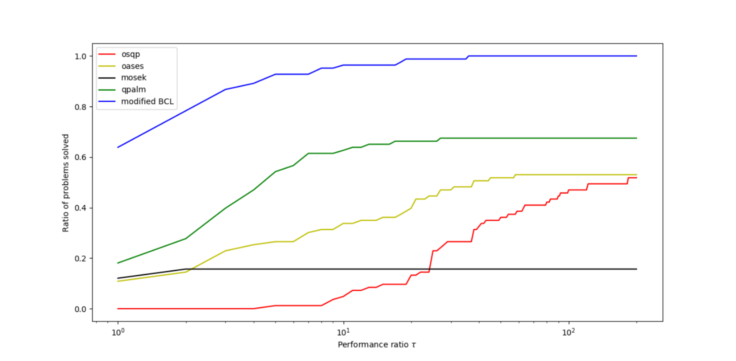

Figure: Performance profiles of our approach, QPALM, OSQP, QPoases, Mosek