Leveraging augmented-Lagrangian techniques for differentiating over infeasible quadratic programs in machine learning

Antoine Bambade[1,2]

[1]Inria and ENS Paris : Willow and Sierra teams
[2]École des Ponts Paris Tech

Joint work with Fabian Schramm, Adrien Taylor, Justin Carpentier

Quadratic programming layer pipeline

More recent literature considers differentiable optimization problems as layers.

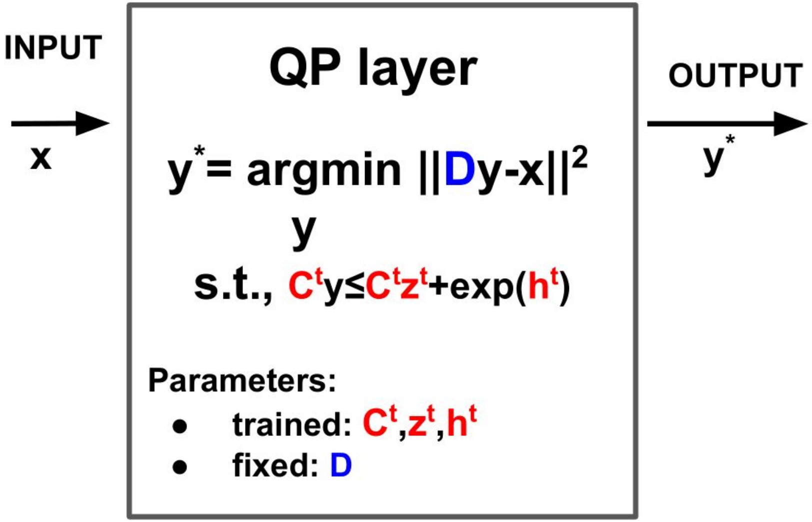


Figure: Example of a Quadratic Programming Layer (with D nonsingular)

QP layers in machine learning

Convex QP layers performs better than a ConvNet for solving Sudokus.

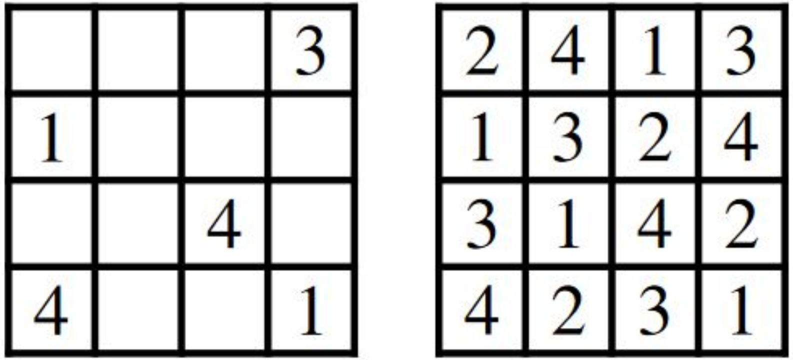


Figure: Example of Sudoku.

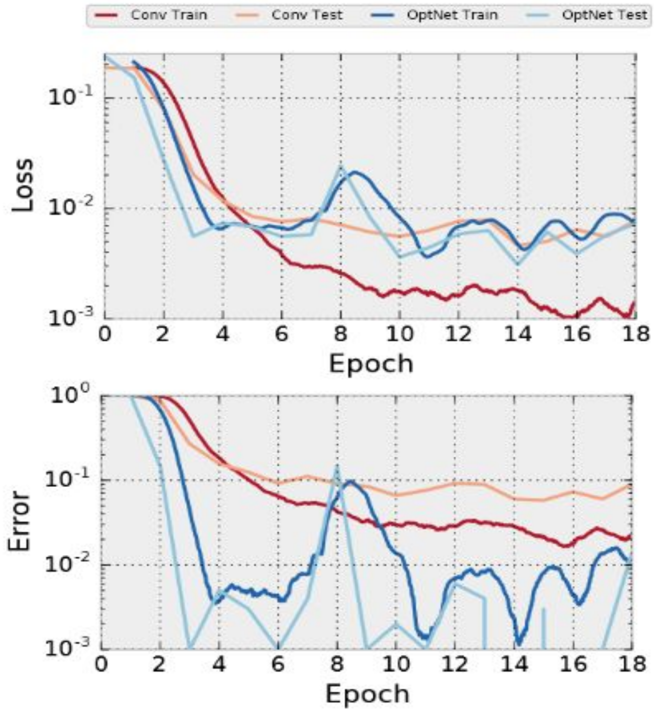


Figure: Training and test plots[1].

[1]B. Amos, Z. Kolter (2021)

QP layers cons: limited trainable architecture

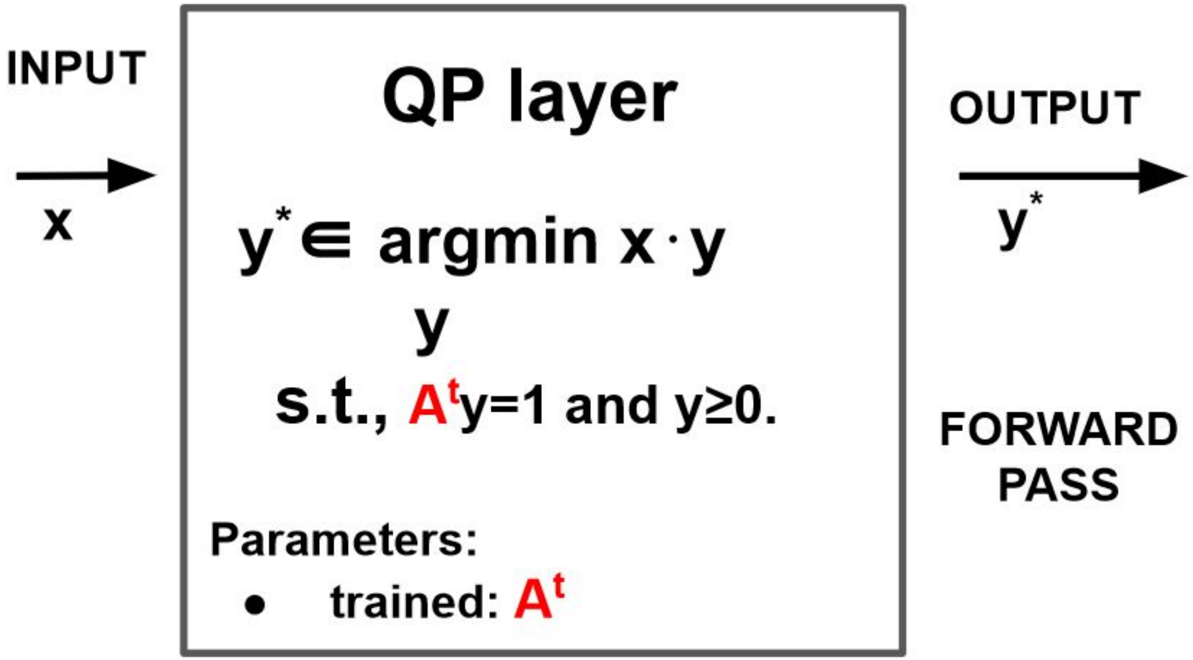
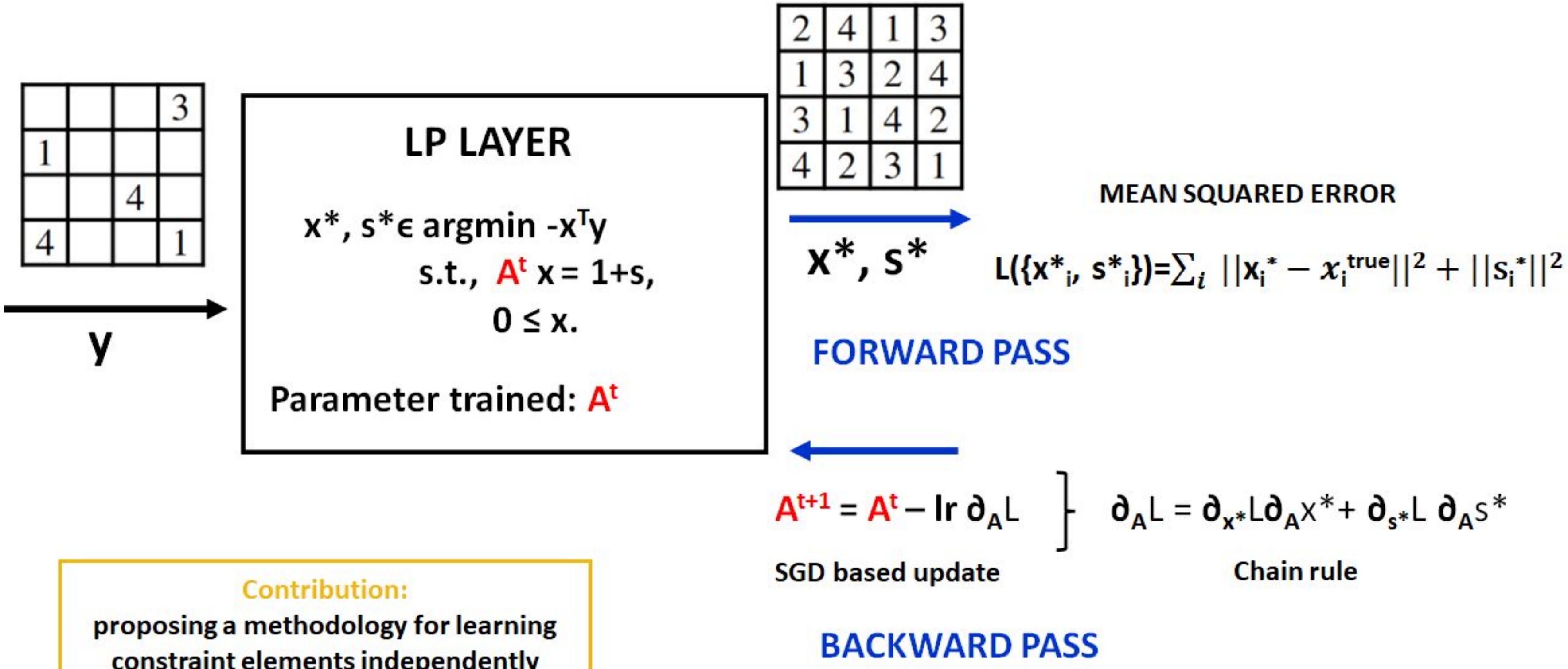


Figure: a LP layer. Nothing guarantees during training that the vector of 1 lies in the range space of A^t.

Solution outline: ideal pipeline



Contribution:
proposing a methodology for learning constraint elements independently

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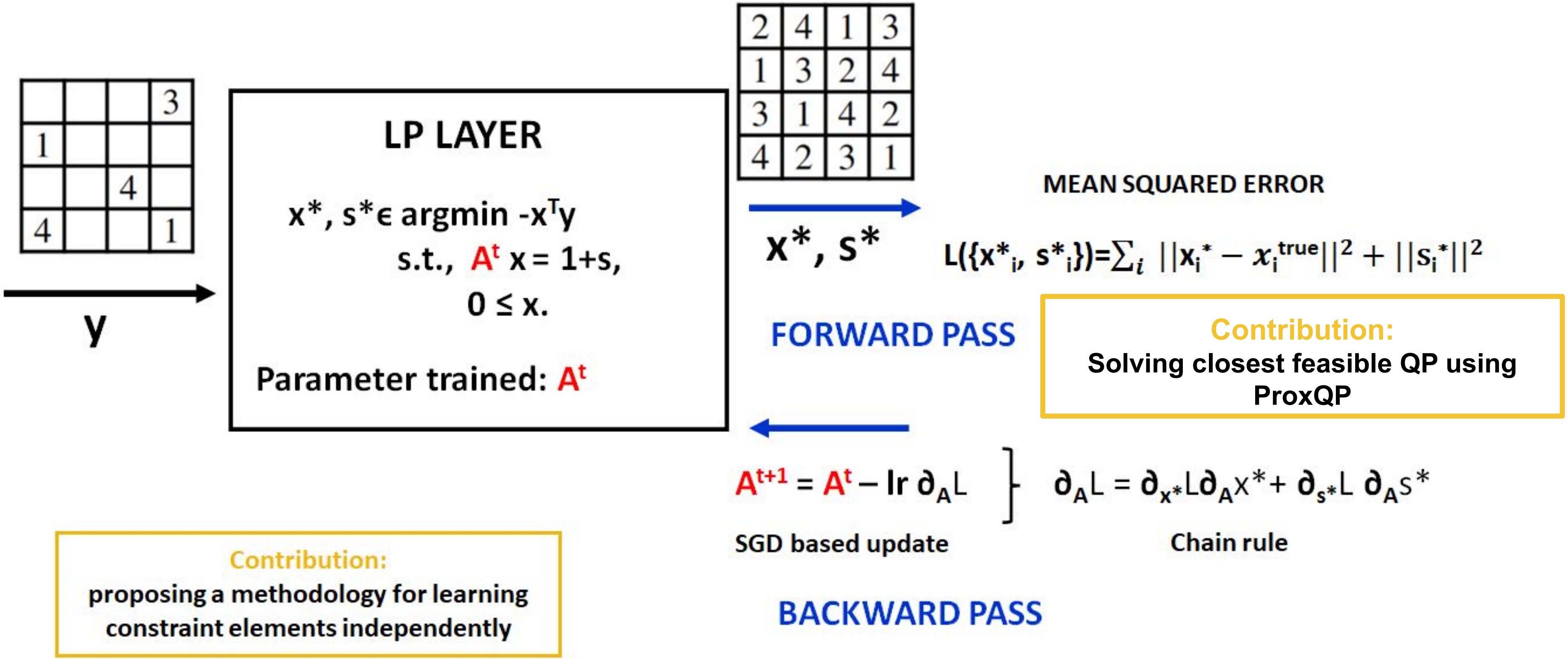
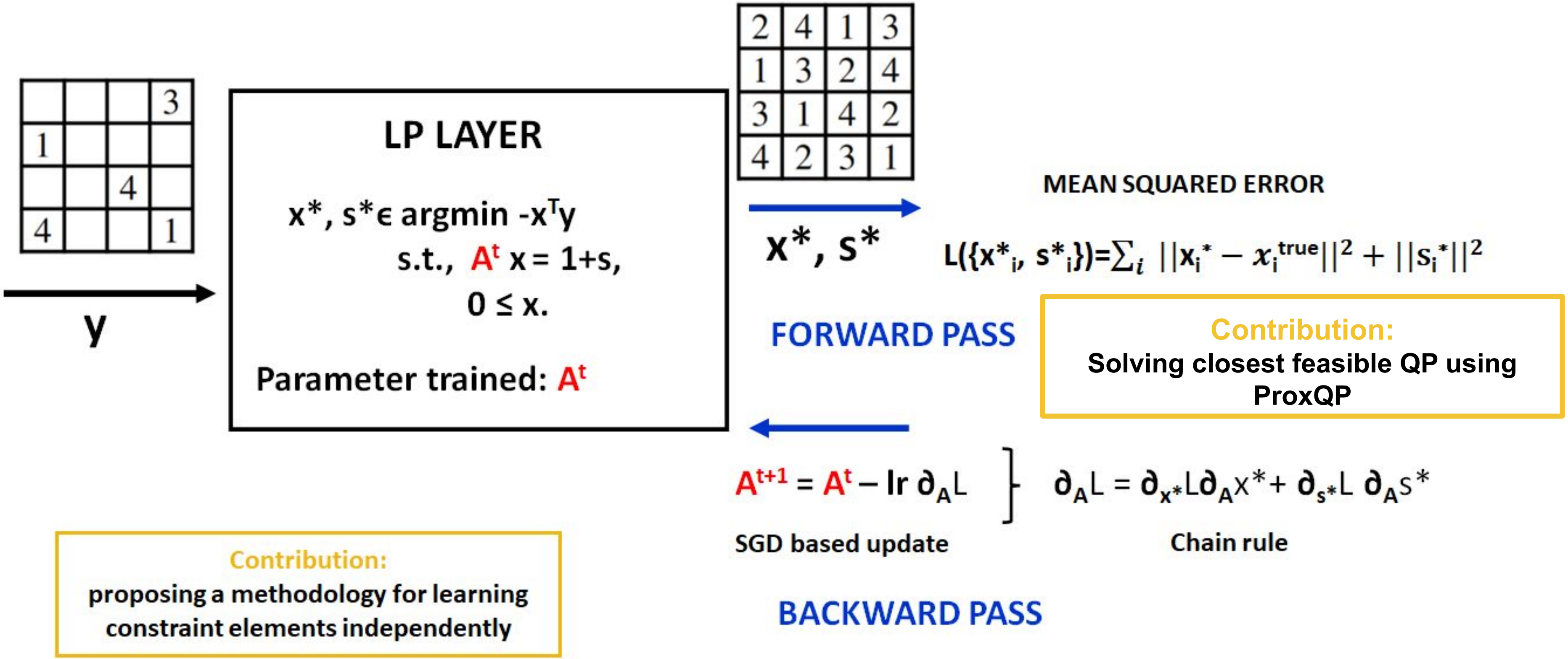


A. Chiche, J-C. Gilbert (2016)



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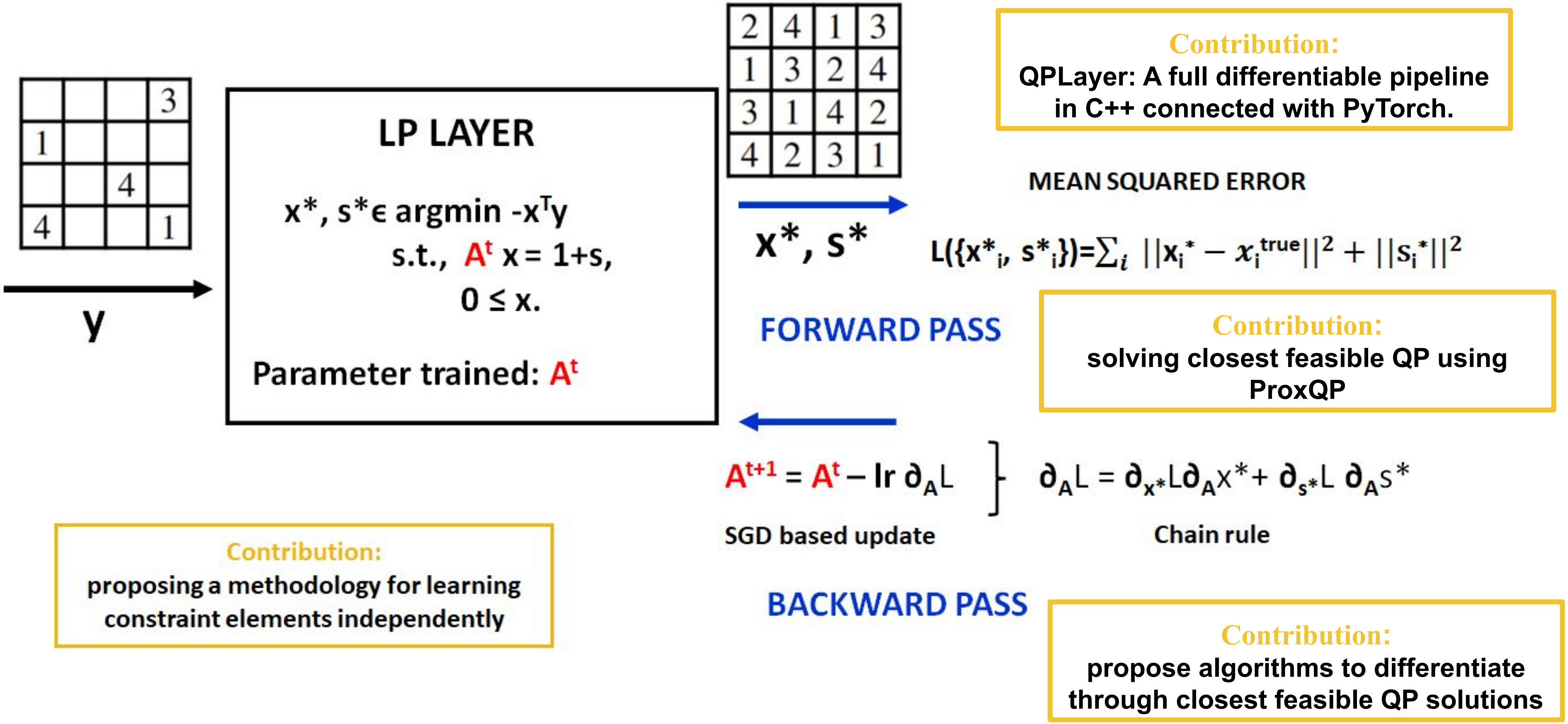

3
1
4
4
1
y
LP LAYER
x*, s*ϵ argmin -x^T y
s.t., A^t x = 1+s,
0 ≤ x.
Parameter trained: A^t
2 4 1 3
1 3 2 4
3 1 4 2
4 2 3 1
x*, s*
FORWARD PASS
Contribution:
QPLayer: A full differentiable pipeline in C++ connected with PyTorch.
MEAN SQUARED ERROR
L({x*_i, s*_i})=∑_i ||x_i* − x_i^true||^2 + ||s_i*||^2
Contribution:
solving closest feasible QP using ProxQP
A^{t+1} = A^t – lr ∂_A L
SGD based update
∂_A L = ∂_{x*} L ∂_A x* + ∂_{s*} L ∂_A s*
Chain rule
BACKWARD PASS
Contribution:
proposing a methodology for learning constraint elements independently
Contribution:
propose algorithms to differentiate through closest feasible QP solutions

Software contribution

ProxSuite

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The backward pass: differentiating closest QP solutions

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A classical technique: **the Implicit Function Theorem.**

The backward pass: differentiating closest QP solutions

$$s^\star(\theta) = \arg \min_{s \in \mathbb{R}^{n_i}} \frac{1}{2}\|s\|_2^2$$

$$\text{s.t. } x^\star(\theta), z^\star(\theta) \in \arg \min_{x \in \mathbb{R}^n} \max_{z \in \mathbb{R}_+^{n_i}} L(x, z, s;\ \theta),$$

$$\text{with } L(x, z, s;\ \theta) \triangleq f(x;\ \theta) + z^\top (C(\theta)x - u(\theta) - s).$$

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A classical technique: **the Implicit Function Theorem.**

$$G(x, z, t;\ \theta) \triangleq \begin{bmatrix} \nabla_x f(x;\ \theta) + C(\theta)^\top z \\ C(\theta)x - u(\theta) - t \\ [[t]_- + z]_+ - z \\ C(\theta)^\top [t]_+ \end{bmatrix}$$

The backward pass: differentiating closest QP solutions

$$
\begin{aligned}
s^\star(\theta) =& \arg\min_{s\in\mathbb{R}^{n_i}} \ \tfrac{1}{2}\|s\|_2^2 \\
& \text{s.t. } x^\star(\theta), z^\star(\theta) \in \arg\min_{x\in\mathbb{R}^n} \max_{z\in\mathbb{R}_+^{n_i}} L(x,z,s;\ \theta),
\end{aligned}
$$

$$
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The map is **path-differentiable**.

E. Pauwels et al. (2019)

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Extend the technique for the closest feasible QP solutions.

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Efficient algorithms to solve these problems.

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Numerical benchmark: back to the Sodoku problem.

Convex QP layers performs better than a ConvNet for solving Sudokus.

			3
1			
		4	
4			1

2	4	1	3
1	3	2	4
3	1	4	2
4	2	3	1

Figure: Example of Sudoku.

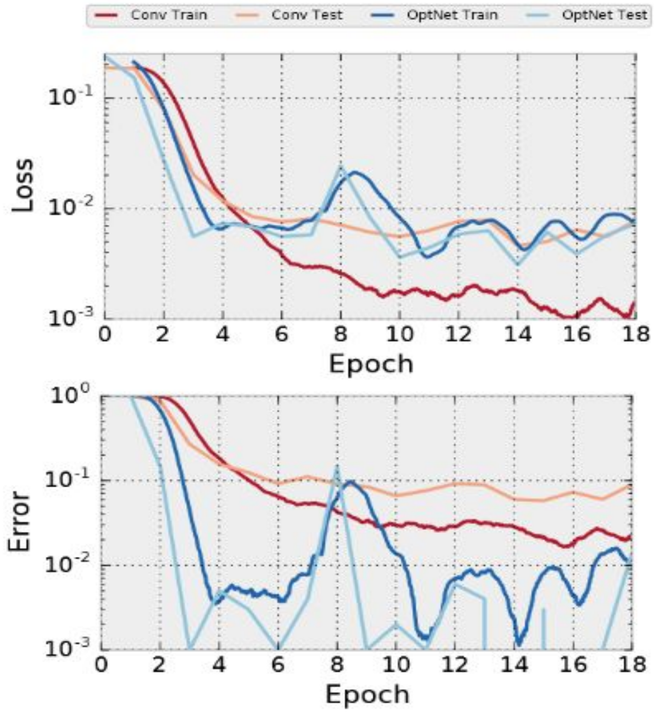

Figure: Training and test plots[1].

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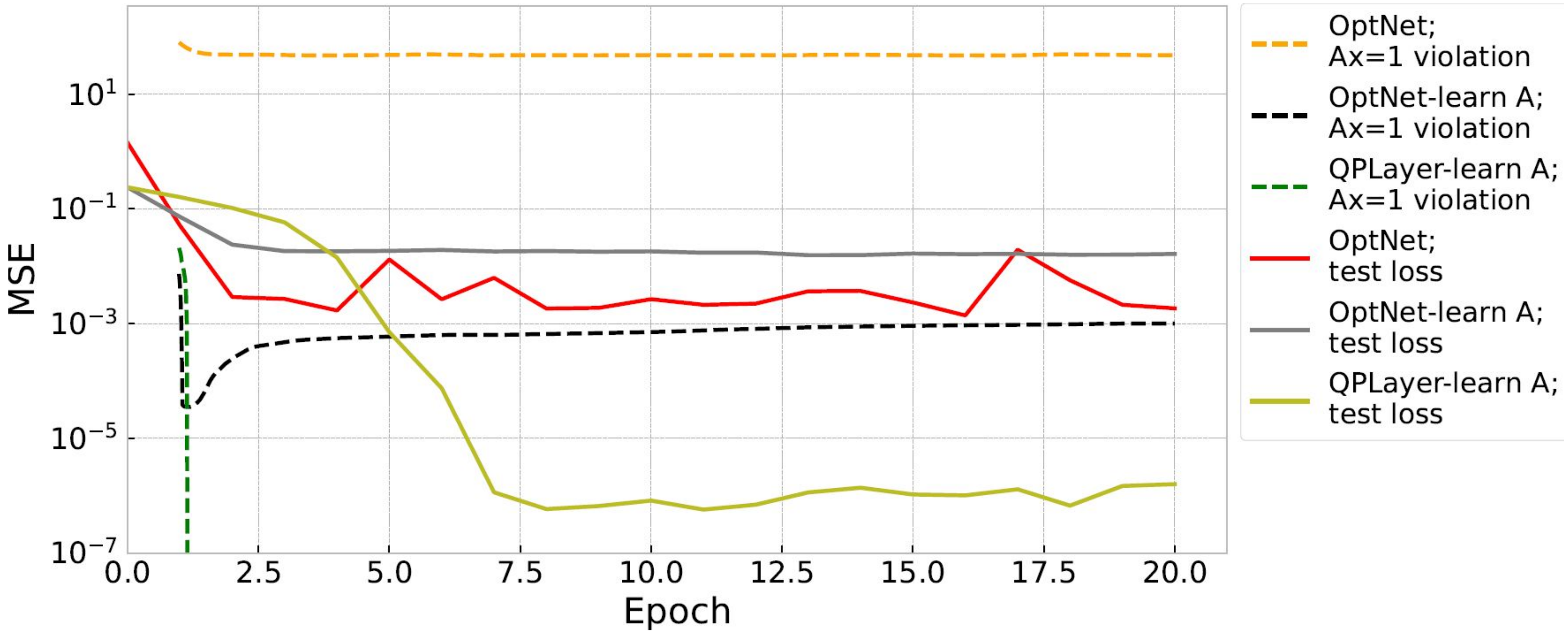
Loss comparison



- **Methodology for learning new QP layers**
  - IFT for closest feasible QPs
  - Extended conservative Jacobians
- **QPlayer: open-source differentiable pipeline**
  - Use Augmented-Lagrangian techniques
  - Connected with PyTorch

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https://github.com/Simple-Robotics/proxsuite

ProxSuite

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Exact Method of Multipliers launched from $\mathbf{z^0}$ with $\sum_k 1/\mu^k$ finite

$$z^{k+1} = z^k - 1/\mu^k s^k, \quad s^k \in \partial\delta(z^{k+1})$$